

Ramification of local fields with imperfect residue fields III

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Abstract

The graded quotients of the logarithmic ramification groups of a local field of mixed characteristic is killed by the residue characteristic. Its characters are described by differential forms.

Let K be a complete discrete valuation field and F be the residue field. We do not assume that F is perfect. We fix a separable closure \bar{K} of K . The residue field \bar{F} of \bar{K} is an algebraic closure of F . In [1, Definition 3.12], we defined a filtration by ramification groups on the absolute Galois group $G_K = \text{Gal}(\bar{K}/K)$ and a logarithmic variant. In this paper, we only consider the logarithmic variant and, by dropping the suffix log in the notation, let $(G_K^r)_{r \in \mathbb{Q}, r > 0}$ denote the decreasing filtration by logarithmic ramification groups. For $r > 0$, we put $G_K^{r+} = \overline{\bigcup_{s>r} G_K^s}$. In [2, Theorem 5.12], we proved that the graded quotient $\text{Gr}^r G_K = G_K^r / G_K^{r+}$ is an abelian group. In this paper, we prove the following.

Theorem 1 *Assume that the residue field F is of characteristic $p > 0$. Then, the abelian group $\text{Gr}^r G_K = G_K^r / G_K^{r+}$ is annihilated by p for every $r > 0$.*

Liang Xiao claims Theorem 1 in [4, Theorem 3.7.3].

Theorem 1 is proved in the equal characteristic case in [3, Corollary 1.27]. Similarly as in the equal characteristic case, it is reduced to the case where the residue field F is of finite type over a perfect subfield k . In this case, an F -vector space $\Omega_F(\log)$ of finite dimension fitting in an exact sequence $0 \rightarrow \Omega_{F/k}^1 \rightarrow \Omega_F(\log) \rightarrow F \rightarrow 0$ is defined (see (1.1)). Let $\text{ord}_{\bar{K}}$ be the valuation of \bar{K} extending the normalized valuation ord_K of K and we put $\mathfrak{m}_{\bar{K}}^r = \{x \in \bar{K} \mid \text{ord}_{\bar{K}} x \geq r\}$ and $\mathfrak{m}_{\bar{K}}^{r+} = \{x \in \bar{K} \mid \text{ord}_{\bar{K}} x > r\}$ and we consider the \bar{F} -vector space $\Theta_{\bar{F}}^{(r)} = \text{Hom}_F(\Omega_F(\log), \mathfrak{m}_{\bar{K}}^r / \mathfrak{m}_{\bar{K}}^{r+})$ as a smooth additive algebraic group over \bar{F} . In [2, (5.12.1)], further a canonical surjection

$$(0.1) \quad \pi_1(\Theta_{\bar{F}}^{(r)})^{\text{ab}} \rightarrow \text{Gr}^r G_K$$

is defined (see (1.6)). Let $\pi_1(\Theta_{\bar{F}}^{(r)})^{\text{alg}}$ denote the quotient of $\pi_1(\Theta_{\bar{F}}^{(r)})^{\text{ab}}$ classifying étale isogenies. Then, $\pi_1(\Theta_{\bar{F}}^{(r)})^{\text{alg}}$ is a profinite abelian group killed by $p = \text{char} F > 0$ and

the character group $\text{Hom}(\pi_1(\Theta_{\bar{F}}^{(r)})^{\text{alg}}, \mathbb{F}_p)$ is canonically identified with the dual space $\text{Hom}_{\bar{F}}(\mathfrak{m}_{\bar{K}}^r/\mathfrak{m}_{\bar{K}}^{r+}, \Omega_F(\log) \otimes \bar{F})$, by pulling-back the Artin-Schreier covering $\mathbf{G}_a \rightarrow \mathbf{G}_a : t \mapsto t^p - t$ by linear form. The main theorem of this paper is the following.

Theorem 2 *We assume that the residue field F is finitely generated over a perfect subfield k of characteristic $p > 0$. Then, the canonical surjection (0.1) factors through the quotient $\pi_1(\Theta_{\bar{F}}^{(r)})^{\text{alg}}$. Consequently, the abelian group $\text{Gr}^r G_K$ is killed by p and there exists a canonical injection*

$$(0.2) \quad \text{Hom}(\text{Gr}^r G_K, \mathbb{F}_p) \rightarrow \text{Hom}_{\bar{F}}(\mathfrak{m}_{\bar{K}}^r/\mathfrak{m}_{\bar{K}}^{r+}, \Omega_F(\log) \otimes \bar{F}).$$

Since Theorem 2 is also proved in the equal characteristic case in [3, Theorem 1.24], we prove it in the mixed characteristic case in this paper. The basic idea of the proof is the same as in the equal characteristic case. However, in the mixed characteristic case, the projections that played a crucial role in the proof in the equal characteristic case are not defined as maps of schemes. They are defined only as an infinitesimal deformation of a morphism of discrete valuation fields in the sense of Definition 2.1. We show that an infinitesimal deformation induces a functor of Galois categories in Section 2 and that they satisfy a transitivity property in Section 3. We prove Theorem 2 in Section 4. In Section 1, we briefly recall fundamental constructions in the definition of filtrations by ramification groups.

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1 Brief review of ramification theory

Let K be a complete discrete valuation field. We assume that K is of characteristic 0 and the residue field F is of characteristic $p > 0$. We assume that F is finitely generated over a perfect subfield k . We define an F -vector space of finite dimension $\Omega_F(\log)$ by

$$(1.1) \quad \Omega_F(\log) = (\Omega_{F/k}^1 \oplus (F \otimes K^\times)) / (d\bar{a} - \bar{a} \otimes a; a \in \mathcal{O}_K^\times),$$

by an abuse of notation because $\Omega_F(\log)$ depends not only on F but also on K . It fits in an exact sequence $0 \rightarrow \Omega_{F/k}^1 \rightarrow \Omega_F(\log) \rightarrow F \rightarrow 0$. For $a \in K^\times$, the image of $1 \otimes a$ is denoted by $d \log a$.

Let \bar{K} be an algebraic closure of K . The residue field \bar{F} of \bar{K} is an algebraic closure of F . Let G_K and G_F be the absolute Galois groups $\text{Gal}(\bar{K}/K)$ and $\text{Gal}(\bar{F}/F)$. We have a canonical surjection $G_K \rightarrow G_F$. Let $(G_K^r)_{r \in \mathbb{Q}, r > 0}$ denote the decreasing filtration by logarithmic ramification groups. For $r > 0$, we put $G_K^{r+} = \overline{\bigcup_{s>r} G_K^s}$. For a finite étale K -algebra L , we say that the log ramification of L is bounded by $r+$ if the natural action of G_K on the finite set $\text{Hom}_K(L, \bar{K})$ factors through the quotient $G_K^{\leq r+} = G_K/G_K^{r+}$. Let $\mathcal{C}_K^{\leq r+}$ denote the category of finite étale K -algebras

of log ramification bounded by $r+$. We identify the category $\mathcal{C}_K^{\leq r+}$ with the category $(G_K^{\leq r+}\text{-Sets})$ of finite sets with continuous action of $G_K^{\leq r+}$ by the natural anti-equivalence defined by the fiber functor attaching $\text{Hom}_K(L, \bar{K})$ to L .

In the following of this section, we assume that $r > 0$ is an integer. Let $\Theta^{(r)}$ denote the F -vector space $\text{Hom}_F(\Omega_F(\log), \mathfrak{m}_K^r/\mathfrak{m}_K^{r+1})$ of finite dimension regarded as a smooth algebraic group over F . We consider a natural action of G_F on $\Theta_{\bar{F}}^{(r)} = \Theta^{(r)} \times_F \bar{F}$. Let $(G_K^{\leq r+}\text{-FEt}/\Theta_{\bar{F}}^{(r)})$ denote the category of finite étale schemes over $\Theta_{\bar{F}}^{(r)}$ with a continuous action of $G_K^{\leq r+} = G_K/G_K^{r+}$ compatible with that of G_F on $\Theta_{\bar{F}}^{(r)}$. We briefly recall the construction of the functor

$$(1.2) \quad X_K^{(r)} : \mathcal{C}_K^{\leq r+} \longrightarrow (G_K^{\leq r+}\text{-FEt}/\Theta_{\bar{F}}^{(r)})$$

in [2] with a slight modification replacing complete local rings by schemes of finite type.

Lemma 1.1 *Let $L = \prod_j L_j$ be a finite étale K -algebra and we put $S = \text{Spec } \mathcal{O}_K$ and $T = \text{Spec } \mathcal{O}_L$. Then, there exists a commutative diagram*

$$(1.3) \quad \begin{array}{ccccc} T & \xrightarrow{i'} & Q_0 & \longleftarrow & E_0 \\ \downarrow & & \downarrow & & \downarrow \\ S & \xrightarrow{i} & P_0 & \longleftarrow & D_0 \end{array}$$

of schemes over the ring $W(k)$ of Witt vectors satisfying the following conditions:

- (1.3.1) *The schemes P_0, Q_0, D_0 and E_0 are smooth over $W(k)$ and $D_0 \subset P_0$ and $E_0 \subset Q_0$ are divisors. The vertical arrows are finite and flat. The left square is cartesian.*
- (1.3.2) *Let $s = \text{Spec } F \in S$ denote the closed point and $t_j = \text{Spec } F_j \in T$ denote the closed points. Then, the maps i and i' induces isomorphisms $\kappa(i(s)) \rightarrow F$ and $\kappa(i'(t_j)) \rightarrow F_j$ of residue fields. The closed subschemes $S \times_{P_0} D_0$ and $T \times_{Q_0} E_0$ are equal to $\text{Spec } F$ and to the reduced part $(T \times_S \text{Spec } F)_{\text{red}} = \coprod_j \text{Spec } F'_j$ respectively. The canonical maps $\Omega_{P_0/W(k)}^1(\log D_0) \otimes F \rightarrow \Omega_F(\log)$ and $\Omega_{Q_0/W(k)}^1(\log E_0) \otimes F'_j \rightarrow \Omega_{F'_j}(\log)$ are isomorphisms for every j . On a neighborhood of $i'(t_j)$, the pull-back $D_0 \times_{P_0} Q_0$ is equal to the divisor $e_j E_0$ where e_j is the ramification index, for every j .*

Proof. We take elements $a_1, \dots, a_n \in \mathcal{O}_K$ such that the images $\bar{a}_1, \dots, \bar{a}_n$ in F form a transcendental basis F over k and that F is a finite separable extension of $k(\bar{a}_1, \dots, \bar{a}_n)$. Let A_0 be the henselization of the subring $W(k)[a_1, \dots, a_n] \subset \mathcal{O}_K$ at the prime ideal (p) and K_0 be the fraction field of the completion of the henselian discrete valuation ring A_0 .

Then K is a finite separable extension of K_0 . Let $K_1 \subset K$ be the maximum unramified subextension over K_0 . Then, there exist unique finite flat normal A_0 -subalgebras $A_1 \subset A$ of \mathcal{O}_K such that the natural maps $A \otimes_{A_0} \mathcal{O}_{K_0} \rightarrow \mathcal{O}_K$ and $A_1 \otimes_{A_0} \mathcal{O}_{K_0} \rightarrow \mathcal{O}_{K_1}$ are isomorphisms.

We take a prime element π of A and let $f \in A_1[t]$ be the minimal polynomial. Let $A_1\{t\}$ be the henselization at the maximal ideal (p, t) . Then, we obtain an isomorphism $A_1\{t\}/(f) \rightarrow A$. It induces an isomorphism $A_1\{t\}/(f, t) \rightarrow F$.

We may assume L is a finite separable extension of K . Similarly, there exists a unique finite flat normal A_0 -subalgebra B of \mathcal{O}_L such that the natural map $B \otimes_{A_0} \mathcal{O}_{K_0} \rightarrow \mathcal{O}_L$ is an isomorphism.

Let F' be the residue field of L . We take elements $b_1, \dots, b_n \in B$ such that the images $\bar{b}_1, \dots, \bar{b}_n$ in F' form a transcendental basis F' over k and that F' is a finite separable extension of $k(\bar{b}_1, \dots, \bar{b}_n)$. Let B_0 be the henselization of the subring $W(k)[b_1, \dots, b_n] \subset \mathcal{O}_L$ at the prime ideal (p) . Then, we obtain $L_0 \subset L_1 \subset L$ and $B_0 \subset B_1 \subset B$ as above. We take a prime element π' of B and let $g \in B_1[t']$ be the minimal polynomial. Then, we obtain an isomorphism $B_1\{t'\}/(g) \rightarrow B$. It induces an isomorphism $B_1\{t'\}/(g, t') \rightarrow F'$.

Since A_1 is essentially smooth over $W(k)$, there exists a map $A_1 \rightarrow B_1\{t'\}$ over $W(k)$ lifting the composition $A_1 \rightarrow A \rightarrow B = B_1\{t'\}/(g)$. We put $\pi = u\pi'^e$ for $u \in B^\times$ and take a lifting $\tilde{u} \in B_1\{t'\}^\times$. We extend the map $A_1 \rightarrow B_1\{t'\}$ to a map $A_1\{t\} \rightarrow B_1\{t'\}$ by sending t to $\tilde{u} \cdot t'^e$. Thus, we obtain a commutative diagram

$$(1.4) \quad \begin{array}{ccccc} B & \longleftarrow & B_1\{t'\} & \xrightarrow{t' \mapsto 0} & B_1 \\ \uparrow & & \uparrow & & \uparrow \\ A & \longleftarrow & A_1\{t\} & \xrightarrow{t \mapsto 0} & A_1 \end{array}$$

of $W(k)$ -algebras.

We show that the left square gives an isomorphism $A \otimes_{A_1\{t\}} B_1\{t'\} \rightarrow B$. Since the maximal ideal of A_1 is generated by the image of f , the maximal ideal of B_1 is also generated by the image of f . Hence, the image of f in $B_1\{t'\}$ is not in the square of the maximal ideal and we have $(f) = (g)$ as ideals of $B_1\{t'\}$. Therefore the map $A \otimes_{A_1\{t\}} B_1\{t'\} = B_1\{t'\}/(f) \rightarrow B = B_1\{t'\}/(g)$ is an isomorphism. Consequently, the map $A_1\{t\} \rightarrow B_1\{t'\}$ is finite flat. Since the question is étale local on neighborhoods of the images of S and T , we deduce a diagram (1.3) satisfying the conditions (1.3.1) and (1.3.2) from the diagram (1.4). \blacksquare

We define a modification $P^{(r)}$ of the scheme $P_0 \times_{W(k)} S$ as follows. We take a blow-up of $P_0 \times_{W(k)} S$ at $D_0 \times_{W(k)} \text{Spec } F$ and define a scheme P over S to be the complement of the union of the proper transforms of $P_0 \times_{W(k)} \text{Spec } F$ and $D_0 \times_{W(k)} S$. The map $S \rightarrow P_0$ induces a section $S \rightarrow P$. We regard $S_r = \text{Spec } \mathcal{O}_K/\mathfrak{m}_K^r$ as a closed subscheme of P by the composition $S_r \rightarrow S \rightarrow P$. We consider the blow-up of P at the closed subscheme S_r and define $P^{(r)}$ to be the complement of the proper transform of the closed fiber $P \times_S \text{Spec } F$. The schemes P and $P^{(r)}$ are smooth over S .

More concretely, the schemes P and $P^{(r)}$ are described as follows. Assume $P_0 = \text{Spec } A_0$ is affine and the divisor D_0 is defined by $t \in A_0$. The image $\pi \in \mathcal{O}_K$ of t by the map $A_0 \rightarrow \mathcal{O}_K$ corresponding to $S \rightarrow P_0$ is a uniformizer of K . Then, we have $P = \text{Spec } A$ for $A = A_0 \otimes_{W(k)} \mathcal{O}_K[U^{\pm 1}]/(Ut - \pi)$. Let I be the kernel of the surjection

$A \rightarrow \mathcal{O}_K$ induced by $A_0 \rightarrow \mathcal{O}_K$ and $U \mapsto 1$. Then, we have $P^{(r)} = \text{Spec } A^{(r)}$ for $A^{(r)} = A[I/\pi^r] \subset A[1/\pi]$.

The closed fiber $P_F^{(r)} = P^{(r)} \times_S \text{Spec } F$ is canonically identified with the affine space $\Theta^{(r)}$ as follows. The canonical map $\Omega_{P_0/S}^1 \otimes_{\mathcal{O}_{P_0}} \mathcal{O}_P \rightarrow \Omega_{P/S}^1$ is uniquely extended to an isomorphism $\Omega_{P_0/S}^1(\log D_0) \otimes_{\mathcal{O}_{P_0}} \mathcal{O}_P \rightarrow \Omega_{P/S}^1$. Let $\mathcal{I} \subset \mathcal{O}_P$ denote the ideal sheaf defining the closed subscheme $S \subset P$. Then, the closed fiber $P_F^{(r)}$ is canonically identified with the F -vector space $\text{Hom}_F(\mathcal{I}/\mathcal{I}^2 \otimes F, \mathfrak{m}_K^r/\mathfrak{m}_K^{r+1})$ regarded as an affine space. By the isomorphisms $\Omega_{P_0/S}^1(\log D_0) \otimes_{\mathcal{O}_{P_0}} \mathcal{O}_P \rightarrow \Omega_{P/S}^1$, $\mathcal{I}/\mathcal{I}^2 \rightarrow \Omega_{P/S}^1 \otimes_{\mathcal{O}_P} \mathcal{O}_S$ and $\Omega_{P_0/S}^1(\log D_0) \otimes F \rightarrow \Omega_F(\log)$, we obtain a canonical isomorphism

$$(1.5) \quad P_F^{(r)} \rightarrow \Theta^{(r)}.$$

Let $Q_S^{(r)}$ be the normalization of the base change $Q_0 \times_{P_0} P_S^{(r)}$ and $Q_F^{(r)}$ be the closed fiber. Then, by the description of log ramification groups [2, Section 5.1], the log ramification of L is bounded by $r+$ if and only if the finite map $Q_F^{(r)} \rightarrow \Theta_F^{(r)}$ is étale. Further, it is shown in [2, Lemma 5.10] that, if the log ramification of L is bounded by $r+$, the finite étale scheme $Q_F^{(r)} \rightarrow \Theta_F^{(r)}$ with the natural action of G_K is independent of the choice of a diagram (1.3) and is well-defined up to unique isomorphism. The functor $X_K^{(r)}: \mathcal{C}_K^{\leq r+} \rightarrow (G_K^{\leq r+}\text{-FEt}/\Theta_F^{(r)})$ (1.2) is defined by attaching $Q_F^{(r)}$ to L . The composition of $X_K^{(r)}: \mathcal{C}_K^{\leq r+} \rightarrow (G_K^{\leq r+}\text{-FEt}/\Theta_F^{(r)})$ with the fiber functor $F_0: (G_K^{\leq r+}\text{-FEt}/\Theta_F^{(r)}) \rightarrow (G_K^{\leq r+}\text{-Sets})$ at the origin $0 \in \Theta_F^{(r)}$ recovers the natural equivalence of categories $\mathcal{C}_K^{\leq r+} \rightarrow (G_K^{\leq r+}\text{-Sets})$.

Further, it is shown in [2, Theorem 5.12] that, for a finite étale K -algebra L of log ramification bounded by $r+$, the finite étale covering $X_K^{(r)}(L) \rightarrow \Theta_F^{(r)}$ is trivialized by a universal abelian covering $\Theta_F^{(r)\text{ab}}$. Thus, forgetting the Galois action on $X_K^{(r)}(L)$ and taking the fiber functor at the origin $0 \in \Theta_F^{(r)}$, we obtain a group homomorphism $\pi_1(\Theta_F^{(r)})^{\text{ab}} \rightarrow G_K^{\leq r+}$ defined by the functor $X_K^{(r)}$. It induces a canonical surjection

$$(1.6) \quad \pi_1(\Theta_F^{(r)})^{\text{ab}} \rightarrow \text{Gr}^r G_K$$

[2, (5.12.1)]. It is compatible with the G_K -action defined by the actions of G_F on $\Theta_F^{(r)}$ and the conjugate action on $\text{Gr}^r G_K$. Since the inertia group $I = \text{Ker}(G_K \rightarrow G_F)$ acts trivially on $\Theta_F^{(r)}$, it follows that $\text{Gr}^r G_K$ is a central subgroup of I/G_K^{r+} .

2 Infinitesimal deformation

Let K and K' be complete discrete valuation fields. We say that a morphism $f: K \rightarrow K'$ of fields is an extension of complete discrete valuation fields if it induces a flat local morphism $\mathcal{O}_K \rightarrow \mathcal{O}_{K'}$, also denoted by f by abuse of notation. The integer $e > 0$ characterized by $f(\mathfrak{m}_K)\mathcal{O}_{K'} = \mathfrak{m}_{K'}^e$ is called the ramification index of f .

Definition 2.1 Let $f: K \rightarrow K'$ be an extension of complete discrete valuation fields of ramification index e . For an integer $r > 0$, we call the pair (f, ε) with an f -linear morphism $\varepsilon: \Omega_F(\log) \rightarrow \mathfrak{m}_{K'}^{er}/\mathfrak{m}_{K'}^{er+1}$ an infinitesimal deformation of f .

We define a composition of infinitesimal deformations. Let $f: K \rightarrow K', g: K' \rightarrow K''$ be extensions of complete discrete valuation fields of ramification indices e, e' and $\varepsilon: \Omega_F(\log) \rightarrow \mathfrak{m}_{K'}^{er}/\mathfrak{m}_{K'}^{er+1}$ and $\varepsilon': \Omega_{F'}(\log) \rightarrow \mathfrak{m}_{K''}^{ee'r}/\mathfrak{m}_{K''}^{ee'r+1}$ be f -linear and g -linear morphisms. Let $g_*: \mathfrak{m}_{K'}^{er}/\mathfrak{m}_{K'}^{er+1} \rightarrow \mathfrak{m}_{K''}^{ee'r}/\mathfrak{m}_{K''}^{ee'r+1}$ be the map induced by g and $f_*: \Omega_F(\log) \rightarrow \Omega_{F'}(\log)$ be the map induced by f . Then, we define the composition $(g, \varepsilon') \circ (f, \varepsilon)$ to be $(g \circ f, g_* \circ \varepsilon + \varepsilon' \circ f_*)$.

Let $f: K \rightarrow K'$ be an extension of complete discrete valuation field of ramification index e . Let $r > 0$ an integer and we put $r' = er$. Let $\varepsilon: \Omega_{\mathcal{O}_K}(\log) \rightarrow \mathfrak{m}_{K'}^{r'}/\mathfrak{m}_{K'}^{r'+1}$ be an f -linear morphism. For an infinitesimal deformation (f, ε) of f , we define a functor

$$(2.1) \quad f_{\varepsilon*}: \mathcal{C}_K^{\leq r+} \rightarrow \mathcal{C}_{K'}^{\leq r'+}.$$

We take separable closures $K \subset \bar{K}, K' \subset \bar{K}'$ and an embedding $\bar{f}: \bar{K} \rightarrow \bar{K}'$. The residue fields \bar{F}, \bar{F}' of \bar{K}, \bar{K}' are algebraic closures of F and of F' . By [1, Proposition 3.15 (3)], the map $f^*: G_{K'} \rightarrow G_K$ induces $G_{K'}^{\leq er+} \rightarrow G_K^{\leq r+}$. The embedding $\bar{f}: \bar{K} \rightarrow \bar{K}'$ induces $\bar{f}: \bar{F} \rightarrow \bar{F}'$ and defines a commutative diagram

$$(2.2) \quad \begin{array}{ccc} G_{K'}^{\leq r'+} & \xrightarrow{f^*} & G_K^{\leq r+} \\ \downarrow & & \downarrow \\ G_{F'} & \longrightarrow & G_F \end{array}$$

We recalled the definition of a functor $X_K^{(r)}: \mathcal{C}_K^{\leq r+} \rightarrow (G_K^{\leq r+}\text{-FEt}/\Theta_F^{(r)})$ (1.2) in Section 1. The map ε defines a geometric point $\bar{\varepsilon}: \bar{F}' \rightarrow \Theta_{\bar{F}}^{(r)}$. The map $\bar{\varepsilon}: \bar{F}' \rightarrow \Theta_{\bar{F}}^{(r)}$ is compatible with the morphism $G_{F'} \rightarrow G_F$. By the commutative diagram (2.2), the fiber functor $F_{\bar{\varepsilon}}$ defines a functor $(G_K^{\leq r+}\text{-FEt}/\Theta_F^{(r)}) \rightarrow (G_{K'}^{\leq r'+}\text{-Sets}) \simeq \mathcal{C}_{K'}^{\leq r'+}$. We define a functor $f_{\varepsilon*}$ as the composition

$$(2.3) \quad F_{\bar{\varepsilon}} \circ X_K^{(r)}: \mathcal{C}_K^{\leq r+} \rightarrow (G_K^{\leq r+}\text{-FEt}/\Theta_F^{(r)}) \rightarrow \mathcal{C}_{K'}^{\leq r'+}.$$

To describe a morphism $f_{\varepsilon}^*: G_{K'}^{\leq r'+} \rightarrow G_K^{\leq r+}$ corresponding to the functor $f_{\varepsilon*}$, we introduce a terminology. Let G and G' be groups and $C \subset G$ be a central subgroup. For morphisms of groups $\varphi: G' \rightarrow G$ and $\psi: G' \rightarrow C$, we call the morphism $\varphi_{\psi}: G' \rightarrow G$ defined by $\varphi_{\psi}(g) = \varphi(g)\psi(g)$ for $g \in G'$, the deformation of $\varphi: G' \rightarrow G$ by $\psi: G' \rightarrow C$.

We consider the composition

$$(2.4) \quad G_{K'}^{\leq r'+} \rightarrow G_{F'}^{\text{ab}} \xrightarrow{\varepsilon_*} \pi_1(\Theta_{\bar{F}}^{(r)})^{\text{ab}} \xrightarrow{(1.6)} \text{Gr}^r G_K \subset G_K^{\leq r+} = G_K/G_K^{r+}.$$

As is remarked after (1.6), the subgroup $\text{Gr}^r G_K \subset G_K^{\leq r+}$ is a central subgroup if the residue field F is separably closed.

Lemma 2.2 *Assume that the residue field F is separably closed. Then, the functor $f_{\varepsilon*}: \mathcal{C}_K^{\leq r+} \rightarrow \mathcal{C}_{K'}^{\leq r'+}$ is compatible with the deformation $f_{\varepsilon*}: G_{K'}^{\leq r'+} \rightarrow G_K^{\leq r+}$ of $f^*: G_{K'}^{\leq r'+} \rightarrow G_K^{\leq r+}$ by $\varepsilon_*: G_{K'}^{\leq r'+} \rightarrow \mathrm{Gr}^r G_K \subset G_K^{\leq r+}$ (2.4).*

Proof. We take a lifting $\mathrm{Spec} \bar{F}' \rightarrow \Theta_{\bar{F}}^{(r)\mathrm{ab}}$ to a universal abelian covering of the geometric point $\bar{\varepsilon}: \mathrm{Spec} \bar{F}' \rightarrow \Theta_{\bar{F}}^{(r)}$ and consider the bijection

$$(2.5) \quad X_K^{(r)}(L) \times_{\Theta_{\bar{F}}^{(r)}} \mathrm{Spec} \bar{F}' \rightarrow \pi_0(X_K^{(r)}(L) \times_{\Theta_{\bar{F}}^{(r)}} \Theta_{\bar{F}}^{(r)\mathrm{ab}})$$

of finite sets for a finite étale K -algebra L of ramification bounded by $r+$. By the definition of the functor $f_{\varepsilon*}$, the finite $G_{K'}^{\leq r'+}$ -set $f_{\varepsilon*}(L)$ is defined as $X_K^{(r)}(L) \times_{\Theta_{\bar{F}}^{(r)}} \mathrm{Spec} \bar{F}'$. The bijection (2.5) is compatible with the map $(f^*, \varepsilon_*): G_{K'}^{\leq r'+} \rightarrow G_K^{\leq r+} \times \pi_1(\Theta_{\bar{F}})^{\mathrm{ab}}$. By the definition of the canonical map (1.6), the action of $\pi_1(\Theta_{\bar{F}})^{\mathrm{ab}}$ on the finite set $\pi_0(X_K^{(r)}(L) \times_{\Theta_{\bar{F}}^{(r)}} \Theta_{\bar{F}}^{(r)\mathrm{ab}})$ is the same as that induced from the action of $G_K^{\leq r+}$ by (1.6). Thus the assertion follows. \blacksquare

If we choose a morphism of fiber functors, the functor $f_{\varepsilon*}: \mathcal{C}_K^{\leq r+} \rightarrow \mathcal{C}_{K'}^{\leq r'+}$ induces a morphism of groups $f_{\varepsilon}^*: G_{K'}^{\leq r'+} \rightarrow G_K^{\leq r+}$. Without choosing a morphism of fiber functors, it is still well-defined up to conjugate. Hence, for a representation V of $G_K^{\leq r+}$ the restriction $\mathrm{Res}_{f,\varepsilon} V$ is well-defined up to an isomorphism as a representation of $G_{K'}^{\leq r'+}$.

Corollary 2.3 *Assume that the residue field F is separably closed. Let V be a representation of $G_K^{\leq r+}$ such that the restriction to $\mathrm{Gr}^r G_K$ is a character χ . Let $\varepsilon^*(\chi)$ denote the character of $G_{K'}^{\leq r'+}$ defined as the pull-back by $\varepsilon_*: G_{K'}^{\leq r'+} \rightarrow \mathrm{Gr}^r G_K$ (2.4).*

Then, we have an isomorphism

$$(2.6) \quad \mathrm{Res}_{f,\varepsilon} V \rightarrow \mathrm{Res}_f V \otimes \varepsilon^*(\chi)$$

of representations of $G_{K'}^{\leq r'+}$.

3 Transitivity

Let $f: K \rightarrow K'$ be an extension of complete discrete valuation fields. We say that K' is a smooth extension of K if the ramification index is 1 and if the residue field F' of K' is a finitely generated separable extension of the residue field F of K .

Let $f: K \rightarrow K'$ be a smooth extension of complete discrete valuation field and let $\varepsilon: \Omega_F(\log) \rightarrow \mathfrak{m}_{K'}^r / \mathfrak{m}_{K'}^{r+1}$ be an f -linear map. We consider the dual

$$\mathrm{Hom}_{F'}(\Omega_{F'}(\log), \mathfrak{m}_{K'}^r / \mathfrak{m}_{K'}^{r+1}) \rightarrow \mathrm{Hom}_F(\Omega_F(\log), \mathfrak{m}_K^r / \mathfrak{m}_K^{r+1}) \otimes_F F'$$

of the map $\Omega_F(\log) \otimes_F F' \rightarrow \Omega_{F'}(\log)$ induced by f . We also consider the translation $+\varepsilon$ as a morphism $\Theta_{F'}^{(r)} = \mathrm{Hom}_{F'}(\Omega_{F'}(\log), \mathfrak{m}_{K'}^r / \mathfrak{m}_{K'}^{r+1}) \rightarrow \Theta_{F'}^{(r)}$. Their composition defines a morphism of schemes $f_{\varepsilon}^*: \Theta_{F'}^{(r)} \rightarrow \Theta_{\bar{F}}^{(r)}$ compatible with $G_{F'} \rightarrow G_F$ and hence the pull-back functor $f_{\varepsilon*}: (G_K^{\leq r+}\text{-FEt}/\Theta_{\bar{F}}^{(r)}) \rightarrow (G_{K'}^{\leq r'+}\text{-FEt}/\Theta_{\bar{F}'}^{(r)})$.

Proposition 3.1 *Assume $f: K \rightarrow K'$ is smooth and consider the diagram*

$$\begin{array}{ccc} \mathcal{C}_K^{\leq r+} & \xrightarrow{f_{\varepsilon*}} & \mathcal{C}_{K'}^{\leq r+} \\ X_K^{(r)} \downarrow & & \downarrow X_{K'}^{(r)} \\ (G_K^{\leq r+}\text{-FEt}/\Theta_F^{(r)}) & \xrightarrow{f_{\varepsilon*}} & (G_{K'}^{\leq r+}\text{-FEt}/\Theta_{F'}^{(r)}) \end{array}$$

of functors. Then, there exists an isomorphism

$$(3.1) \quad f_{\varepsilon*} \circ X_K^{(r)} \rightarrow X_{K'}^{(r)} \circ f_{\varepsilon*}$$

of functors.

Proof. We regard $\varepsilon: \Omega_F(\log) \rightarrow \mathfrak{m}_{K'}^r/\mathfrak{m}_{K'}^{r+1}$ as an F' -rational point $\varepsilon: \text{Spec } F' \rightarrow \Theta_F^{(r)}$. Let L be a finite étale algebra over K of log ramification bounded by $r+$. We take a cartesian diagram (1.3) over $W(k)$ as in Lemma 1.1. Take a morphism $S' \rightarrow P^{(r)}$ over S lifting $\varepsilon: \text{Spec } F' \rightarrow \Theta_F^{(r)} \subset P^{(r)}$ and consider the composition $S' \rightarrow P^{(r)} \rightarrow P \rightarrow P_0$.

We put $T' = S' \times_{P_0} Q_0$. Since $\bar{Q}_F^{(r)} \rightarrow \bar{P}_F^{(r)} = \Theta_F^{(r)}$ is étale, the base change $Q_0 \times_{P_0} P^{(r)} \rightarrow P^{(r)}$ is étale on the complement of the closed fiber in a neighborhood of the closed fiber. Hence, the K' -algebra $L' = \Gamma(T' \times_{S'} \text{Spec } K', \mathcal{O})$ is étale. The fiber product $T' \times_{Q_0} E_0$ is isomorphic to $S' \times_{P_0} E_0 = (S' \times_{P_0} D_0) \times_{D_0} E_0 = F' \times_F (F \times_{D_0} E_0) = F' \times_F (T \times_{Q_0} E_0)_{\text{red}}$ and is reduced since F' is separable over F . Hence, T' is the spectrum of the integer ring $\mathcal{O}_{L'}$.

By the assumption that K' is smooth over K , there exists a commutative diagram

$$\begin{array}{ccccc} S' & \longrightarrow & P'_0 & \longleftarrow & D'_0 \\ \downarrow & & \downarrow & & \downarrow \\ S & \longrightarrow & P_0 & \longleftarrow & D_0 \end{array}$$

of schemes over $W(k)$ satisfying the following conditions: The vertical arrow $P'_0 \rightarrow P_0$ is smooth, the right square is cartesian, $S' \times_{P'_0} D'_0 = \text{Spec } F'$ and $\Omega_{P'_0/W(k)}^1(\log D'_0) \otimes F' \rightarrow \Omega_{F'}(\log)$ is an isomorphism.

We consider the diagram

$$(3.2) \quad \begin{array}{ccccc} T' & \longrightarrow & Q'_0 & \longleftarrow & E'_0 \\ \downarrow & & \downarrow & & \downarrow \\ S' & \longrightarrow & P'_0 & \longleftarrow & D'_0 \end{array}$$

where the right square is the base change of that of (1.3) by $P'_0 \rightarrow P_0$ and the left square is cartesian. Since $T' \times_{Q'_0} E'_0 = T' \times_{Q_0} E_0 = F' \times_F (T \times_{Q_0} E_0)_{\text{red}} = (T' \times_{Q_0} E_0)_{\text{red}}$, the diagram (3.2) satisfies the conditions corresponding to (1.3.1) and (1.3.2).

We define $Q_{\bar{S}}^{(r)} \rightarrow P_{\bar{S}}^{(r)}$ and $Q_{\bar{S}'}^{(r)} \rightarrow P_{\bar{S}'}^{(r)}$ and we identify $P_{\bar{S}}^{(r)} = \Theta_{\bar{F}}^{(r)}$ and $P_{\bar{S}'}^{(r)} = \Theta_{\bar{F}'}^{(r)}$ as in (1.5). Then, the map $P_{\bar{S}'}^{(r)} \rightarrow P_{\bar{S}}^{(r)} \times_{\bar{S}} \bar{S}'$ induced by $P' \rightarrow P$ is smooth and hence the diagram

$$(3.3) \quad \begin{array}{ccc} Q_{\bar{S}'}^{(r)} & \longrightarrow & Q_{\bar{S}}^{(r)} \\ \downarrow & & \downarrow \\ P_{\bar{S}'}^{(r)} & \longrightarrow & P_{\bar{S}}^{(r)} \end{array}$$

is cartesian. Since the diagram

$$(3.4) \quad \begin{array}{ccc} \Theta_{\bar{F}'}^{(r)} & \xrightarrow{f_{\varepsilon*}} & \Theta_{\bar{F}}^{(r)} \\ \downarrow & & \downarrow \\ P_{\bar{S}'}^{(r)} & \longrightarrow & P_{\bar{S}}^{(r)} \end{array}$$

is cartesian, we obtain a cartesian diagram

$$(3.5) \quad \begin{array}{ccc} X_{K'}^{(r)}(L') & \longrightarrow & X_K^{(r)}(L) \\ \downarrow & & \downarrow \\ \Theta_{\bar{F}'}^{(r)} & \xrightarrow{f_{\varepsilon*}} & \Theta_{\bar{F}}^{(r)} \end{array}$$

compatible with the group homomorphism $G_{K'}^{\leq r+} \rightarrow G_K^{\leq r+}$.

Thus by the definition of the functor $f_{\varepsilon*}$, we obtain an isomorphism $L' \rightarrow f_{\varepsilon*}(L)$. The diagram (3.5) defines an isomorphism $X_{K'}^{(r)}(L') \rightarrow f_{\varepsilon*}X_K^{(r)}(L)$ in the category $(G_{K'}^{\leq r+}\text{-FEt}/\Theta_{\bar{F}'}^{(r)})$. The isomorphism is functorial in L and they define an isomorphism $X_{K'}^{(r)} \circ f_{\varepsilon*} \rightarrow f_{\varepsilon*} \circ X_K^{(r)}$ of functors. \blacksquare

We deduce the following transitivity.

Corollary 3.2 *Let $f: K \rightarrow K'$ and $g: K' \rightarrow K''$ be extensions of complete discrete valuation fields. We assume f is smooth and let e' be the ramification index of K'' over K' . Let $\varepsilon: \Omega_F(\log) \rightarrow \mathfrak{m}_{K'}^r/\mathfrak{m}_{K'}^{r+1}$ and $\varepsilon': \Omega_{F'}(\log) \rightarrow \mathfrak{m}_{K''}^{e'r}/\mathfrak{m}_{K''}^{e'r+1}$ be infinitesimal deformations and we put $(g, \varepsilon') \circ (f, \varepsilon) = (g \circ f, \varepsilon'')$. Then, there exists an isomorphism of functors:*

$$(3.6) \quad (g \circ f)_{\varepsilon''*} \rightarrow g_{\varepsilon'*} \circ f_{\varepsilon*}.$$

Proof. The composition of the morphism (3.1) with the fiber functor $F_{\bar{\varepsilon}'}$ gives a morphism

$$F_{\bar{\varepsilon}'} \circ f_{\varepsilon*} \circ X_K^{(r)} \rightarrow F_{\bar{\varepsilon}'} \circ X_K^{(r)} \circ f_{\varepsilon*} = g_{\varepsilon'*} \circ f_{\varepsilon*}$$

of functors. By the canonical isomorphism $F_{\bar{\varepsilon}'} \circ f_{\varepsilon*} \rightarrow F_{\bar{\varepsilon}''}$, the first term $F_{\bar{\varepsilon}'} \circ f_{\varepsilon*} \circ X_K^{(r)}$ is identified with the functor $(g \circ f)_{\varepsilon''*}$. \blacksquare

4 Proof of Theorem 2

We prove Theorem 2 in the introduction. It is reduced to the case where $r > 0$ is an integer, by considering the base change by log smooth extension as in [3, Lemma 1.22]. We regard an F -vector space V of finite dimension as a smooth additive algebraic group over F and let $\text{pr}_1, \text{pr}_2: V \times V \rightarrow V$ be the projections and $-: V \times V \rightarrow V$ be the subtraction $(x, y) \mapsto y - x$. By Lemma 4.2 below, it suffices to prove the following.

Lemma 4.1 *Let χ be a character of $\text{Gr}^r G_K$ and regard it also as a character of $\pi_1(\Theta_{\bar{F}}^{(r)})^{\text{ab}}$ by the surjection (1.6). Then, we have an equality $\text{pr}_2^* \chi = \text{pr}_1^* \chi \cdot -^* \chi$ of characters of $\pi_1(\Theta_{\bar{F}}^{(r)} \times \Theta_{\bar{F}}^{(r)})^{\text{ab}}$.*

Lemma 4.2 ([3, Lemma 1.23]) *Let F be an algebraically closed field of characteristic $p > 0$ and regard an F -vector space V of finite dimension as a smooth additive algebraic group over F . Let $\pi_1(V)^{\text{alg}}$ be the quotient of the abelian fundamental group $\pi_1(V)^{\text{ab}}$ classifying étale isogenies. Then, for a character χ of $\pi_1(V)^{\text{ab}}$, the following conditions are equivalent:*

- (1) χ factors through the quotient $\pi_1(V)^{\text{alg}}$.
- (2) We have an equality $\text{pr}_2^* \chi = \text{pr}_1^* \chi \cdot -^* \chi$ of characters of $\pi_1(V \times V)^{\text{ab}}$.

To prove Lemma 4.1, we use the geometric construction in Section 1. We consider the smooth scheme $P^{(r)}$ over S and the fiber product $P^{(r)} \times_S P^{(r)}$. The closed fibers $P_F^{(r)}$ and $P_F^{(r)} \times_F P_F^{(r)}$ are identified with $\Theta_F^{(r)}$ and $\Theta_F^{(r)} \times \Theta_F^{(r)}$. Let $\xi \in \Theta_F^{(r)} \subset P^{(r)}$ and $\eta \in \Theta_F^{(r)} \times_F \Theta_F^{(r)} \subset P^{(r)} \times_S P^{(r)}$ be the generic points. Define complete discrete valuation fields K' and K'' to be the fraction fields of the completions of the local rings $\mathcal{O}_{P^{(r)}, \xi}$ and $\mathcal{O}_{P^{(r)} \times_S P^{(r)}, \eta}$ respectively. They are smooth extensions of K . Let $1: K \rightarrow K'$ denote the canonical map and $p_1, p_2: K' \rightarrow K''$ denote the map induced by the projections $P^{(r)} \times_S P^{(r)} \rightarrow P^{(r)}$.

We define infinitesimal deformations $\varepsilon: \Omega_F(\log) \rightarrow \mathfrak{m}_{K'}^r / \mathfrak{m}_{K'}^{r+1}$ and $\varepsilon': \Omega_{F'}(\log) \rightarrow \mathfrak{m}_{K''}^r / \mathfrak{m}_{K''}^{r+1}$. The residue field F' of K' is the function field of $\Theta_F^{(r)}$ and hence is the fraction field of the symmetric algebra $S_F^\bullet \text{Hom}_F(\mathfrak{m}_K^r / \mathfrak{m}_K^{r+1}, \Omega_F(\log))$ of the dual vector space. We define $\varepsilon: \Omega_F(\log) \rightarrow \mathfrak{m}_{K'}^r / \mathfrak{m}_{K'}^{r+1} = F' \otimes_F \mathfrak{m}_K^r / \mathfrak{m}_K^{r+1}$ to be the map

$$\begin{aligned} \Omega_F(\log) &\rightarrow \text{Hom}_F(\mathfrak{m}_K^r / \mathfrak{m}_K^{r+1}, \Omega_F(\log)) \otimes_F \mathfrak{m}_K^r / \mathfrak{m}_K^{r+1} \\ &\subset S_F^\bullet(\text{Hom}_F(\mathfrak{m}_K^r / \mathfrak{m}_K^{r+1}, \Omega_F(\log))) \otimes_F \mathfrak{m}_K^r / \mathfrak{m}_K^{r+1} \subset F' \otimes_F \mathfrak{m}_K^r / \mathfrak{m}_K^{r+1} \end{aligned}$$

where the arrow is the inverse of the isomorphism defined by the evaluation.

Similarly, the residue field F'' of K'' is the fraction field of the symmetric algebra $S_F^\bullet \text{Hom}_F(\mathfrak{m}_K^r / \mathfrak{m}_K^{r+1}, \Omega_F(\log)^{\oplus 2})$. Since K' is a smooth extension of K , we have an exact sequence $0 \rightarrow \Omega_F(\log) \otimes_F F' \rightarrow \Omega_{F'}(\log) \rightarrow \Omega_{F'/F} \rightarrow 0$. Let $\varepsilon': \Omega_{F'}(\log) \rightarrow \mathfrak{m}_{K''}^r / \mathfrak{m}_{K''}^{r+1} = F'' \otimes_F \mathfrak{m}_K^r / \mathfrak{m}_K^{r+1}$ be a p_1 -linear map such that the restriction to $\Omega_F(\log)$ is

$$\begin{aligned} \Omega_F(\log) &\rightarrow \Omega_F(\log)^{\oplus 2} \rightarrow \text{Hom}_F(\mathfrak{m}_K^r / \mathfrak{m}_K^{r+1}, \Omega_F(\log)^{\oplus 2}) \otimes_F \mathfrak{m}_K^r / \mathfrak{m}_K^{r+1} \\ &\subset S_F^\bullet(\text{Hom}_F(\mathfrak{m}_K^r / \mathfrak{m}_K^{r+1}, \Omega_F(\log)^{\oplus 2})) \otimes_F \mathfrak{m}_K^r / \mathfrak{m}_K^{r+1} \subset F'' \otimes_F \mathfrak{m}_K^r / \mathfrak{m}_K^{r+1} \end{aligned}$$

where the first arrow is the map $x \mapsto (-x, x)$ and the second arrow is the inverse of the isomorphism defined by the evaluation.

Finally, we define a morphism $\mu: K' \rightarrow K''$ over K . The subtraction map $\Theta_F^{(r)} \times_F \Theta_F^{(r)} \rightarrow \Theta_F^{(r)}$ is dominant and induces a morphism $F' \rightarrow F''$ over F . We define $\mu: K' \rightarrow K''$ to be a morphism over K lifting the map $F' \rightarrow F''$.

Lemma 4.3 *We have*

$$(4.1) \quad p_2 \circ (1, \varepsilon) = (p_1, \varepsilon') \circ (1, \varepsilon)$$

$$(4.2) \quad \mu \circ (1, \varepsilon) = (p_1, \varepsilon') \circ 1.$$

Proof. They are equalities of deformations of the canonical map $K \rightarrow K''$. Hence, it suffices to prove the equalities of maps $\Omega_F(\log) \rightarrow \mathfrak{m}_{K''}^r / \mathfrak{m}_{K''}^{r+1} = F'' \otimes \mathfrak{m}_K^r / \mathfrak{m}_K^{r+1}$.

For the left hand side of (4.1), it is induced by the map $\Omega_F(\log) \rightarrow \Omega_F(\log)^{\oplus 2}$ sending x to $(0, x)$. For the right hand side of (4.1), it is induced by the sum of the maps $\Omega_F(\log) \rightarrow \Omega_F(\log)^{\oplus 2}$ sending x to $(x, 0)$ and to $(-x, x)$. For both sides of (4.2), they are induced by the map $\Omega_F(\log) \rightarrow \Omega_F(\log)^{\oplus 2}$ sending x to $(-x, x)$. ■

Proof of Lemma 4.1. We may assume that the residue field F is separably closed. Let χ be a character of $\mathrm{Gr}^r G_K$. Let $\xi^*(\chi)$ be the character of $G_{K'}^{\leq r+}$ defined as the composition of χ with

$$G_{K'}^{\leq r+} \xrightarrow{\xi_*} \pi_1(\Theta_{\bar{F}}^{(r)})^{\mathrm{ab}} \xrightarrow{(1.6)} \mathrm{Gr}^r G_K.$$

Since the canonical map $\eta_*: G_{K''}^{\leq r+} \rightarrow \pi_1(\Theta_{\bar{F}}^{(r)} \times \Theta_{\bar{F}}^{(r)})^{\mathrm{ab}}$ is surjective, it suffices to show the equality $p_2^* \xi^*(\chi) = p_1^* \xi^*(\chi) \cdot \mu^* \xi^*(\chi)$ of characters of $G_{K''}^{\leq r+}$.

Since $\mathrm{Gr}^r G_K$ is a central subgroup of $G_K^{\leq r+}$, there exists an irreducible representation V of $G_K^{\leq r+}$ such that the restriction to $\mathrm{Gr}^r G_K$ is the scalar multiplication by the character χ . By Corollary 2.3, we have an isomorphism $\mathrm{Res}_{1, \varepsilon} V \rightarrow \mathrm{Res}_{G_{K'}^{\leq r+}}^{G_K^{\leq r+}} V \otimes \xi^*(\chi)$ of representations of $G_{K'}^{\leq r+}$. By Corollary 3.2 and by (4.1) and (4.2), it induces isomorphisms

$$\begin{aligned} \mathrm{Res}_{G_{K''}^{\leq r+}}^{G_K^{\leq r+}} V \otimes p_2^* \xi^*(\chi) &\rightarrow \mathrm{Res}_{p_1, \varepsilon'} \mathrm{Res}_{G_{K'}^{\leq r+}}^{G_K^{\leq r+}} V \otimes p_1^* \xi^*(\chi), \\ \mathrm{Res}_{G_{K''}^{\leq r+}}^{G_K^{\leq r+}} V \otimes \mu^* \xi^*(\chi) &\rightarrow \mathrm{Res}_{p_1, \varepsilon'} \mathrm{Res}_{G_{K'}^{\leq r+}}^{G_K^{\leq r+}} V \end{aligned}$$

of representations of $G_{K''}^{\leq r+}$. Thus, we obtain an isomorphism

$$\mathrm{Res}_{G_{K''}^{\leq r+}}^{G_K^{\leq r+}} V \otimes p_2^* \xi^*(\chi) \rightarrow \mathrm{Res}_{G_{K''}^{\leq r+}}^{G_K^{\leq r+}} V \otimes p_1^* \xi^*(\chi) \cdot \mu^* \xi^*(\chi).$$

Since $G_{K''}^{\leq r+} \rightarrow G_K^{\leq r+}$ is surjective, this implies an equality $p_2^* \xi^*(\chi) = p_1^* \xi^*(\chi) \cdot \mu^* \xi^*(\chi)$ of characters of $G_{K''}^{\leq r+}$ by Schur's lemma. Thus the assertion is proved. ■

As in [3, Corollary 1.26], Theorem 2 has the following consequence. Let V be an ℓ -adic representation of G_K . Since $P = G_{K,\log}^{0+}$ is a pro- p group, there exists a unique direct sum decomposition $V = \bigoplus_{q \geq 0, q \in \mathbb{Q}} V^{(q)}$ by sub G_K -modules such that the $G_{K,\log}^{r+}$ -fixed part is given by $V^{G_{K,\log}^{r+}} = \bigoplus_{q \geq r} V^{(q)}$. We put $\mathrm{Sw}_K V = \sum_r r \cdot \mathrm{rank} V^{(r)} \in \mathbb{Q}$.

Corollary 4.4

$$\mathrm{Sw}_K V \in \mathbb{Z}\left[\frac{1}{p}\right].$$

Liang Xiao claims a stronger assertion $\mathrm{Sw}_K V \in \mathbb{Z}$ in [4, Theorem 3.5.11].

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